Method of asymptotic expansions

$$\begin{cases} -\operatorname{div}\left(A_\varepsilon \nabla u^\varepsilon\right) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{where}$$

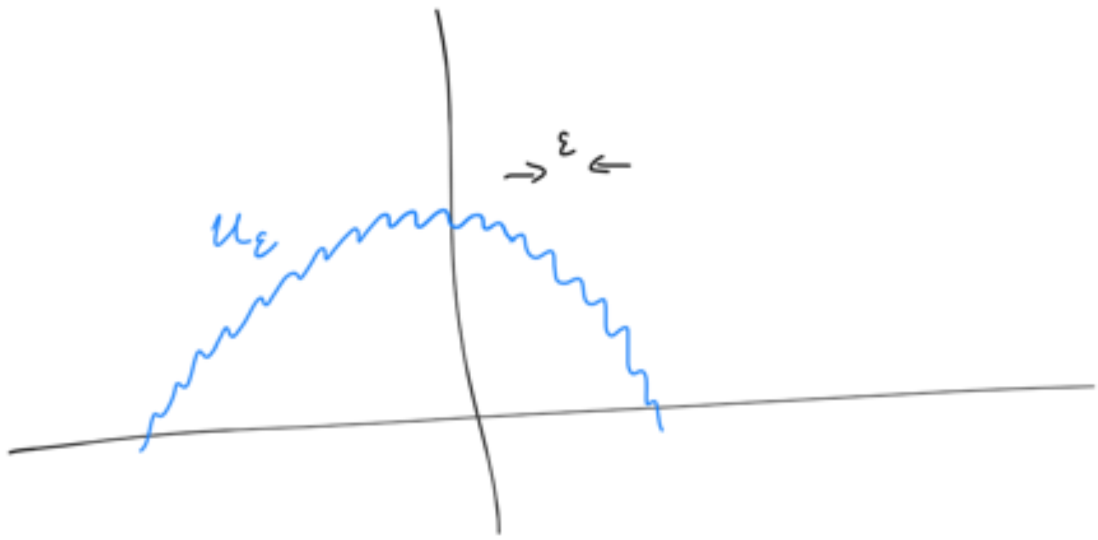
$A_\varepsilon = \left(a_{ij}^\varepsilon\right)$, $\quad a_{ij}^\varepsilon(x) = a_{ij}\left(\frac{x}{\varepsilon}\right)$, $\quad a_{ij}$ 1-periodic,

$$\left.\begin{array}{l} A_\varepsilon(x)\lambda\cdot\lambda \geq \alpha|\lambda|^2 \\ |A_\varepsilon(x)\lambda| \leq \beta|\lambda| \end{array}\right\} \quad \forall \lambda \in \mathbb{R}^N,$$

$0 < \alpha < \beta$.

Coefficients A_ε oscillate on a length scale ε

$\rightsquigarrow$ Expect solution to look like



$\rightsquigarrow$ Asymptotic expansion:

$$u^\varepsilon(x) = u_0\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \tfrac{x}{\varepsilon}\right) + \cdots$$

with $u_j(x,y)$ defined for $x \in \Omega$, $y \in [0,1]^N$ such that

$u_j(x,\cdot)$ is 1-periodic.

Apply $-\operatorname{div}(A_\varepsilon \nabla \cdot)$ to this expansion:

$$f = -\operatorname{div}_x\left(A_\varepsilon \nabla_x\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots\right)\right)$$

$$= -\operatorname{div}_x(A \nabla_x u_0) - \varepsilon^{-1}\operatorname{div}_x(A \nabla_y u_0) - \varepsilon^{-1}\operatorname{div}_y(A \nabla_x u_0)$$

$$- \varepsilon^{-2}\operatorname{div}_y(A \nabla_y u_0)$$

$$- \varepsilon \operatorname{div}_x(A \nabla_x u_1) - \operatorname{div}_x(A \nabla_y u_1) - \operatorname{div}_y(A \nabla_x u_1)$$

$$- \varepsilon^{-1}\operatorname{div}_y(A \nabla_y u_1)$$

$$- \varepsilon^2 \operatorname{div}_x(A \nabla_x u_2) - \varepsilon \operatorname{div}_x(A \nabla_y u_2) - \varepsilon \operatorname{div}_y(A \nabla u_2)$$

$$- \operatorname{div}_y(A \nabla_y u_2)$$

Here: $A = A(y)$

$\rightsquigarrow$ Equations:

(1) $-\operatorname{div}_y(A \nabla_y u_0) = 0$

(2) $-\operatorname{div}_x(A \nabla_y u_0) - \operatorname{div}_y(A \nabla_x u_0) - \operatorname{div}_y(A \nabla_y u_1) = 0$

(3) $-\operatorname{div}_x(A \nabla_x u_0) - \operatorname{div}_y(A \nabla_y u_2) - \operatorname{div}_x(A \nabla_y u_1)$
$\quad - \operatorname{div}_y(A \nabla_x u_1) = f$

Rewrite:

(2') $-\operatorname{div}_y(A \nabla_y u_1) = \left(\operatorname{div}_x A \nabla_y + \operatorname{div}_y A \nabla_x\right) u_0$

(3') $-\operatorname{div}_y(A \nabla_y u_2) = f + \left(\operatorname{div}_x A \nabla_y + \operatorname{div}_y A \nabla_x\right) u_1 + \operatorname{div}_x(A \nabla_x u_0)$

Interpret these as equations in $y \in [0,1]^N$ with periodic boundary conditions; $x \in \Omega$ is parameter.

$\rightsquigarrow$ u_0 indep. of y !

Consider eq. (2'):

u_0 const. in $y \Rightarrow \nabla_y u_0 = 0$ We get:

$$\left.\begin{aligned} -\operatorname{div}_y(A \nabla_y u_1) &= \operatorname{div}_y(A \nabla_x u_0) \\ u_1 \quad & \text{1-periodic} \end{aligned}\right\}$$

Solvability condition: $\displaystyle\int_{[0,1]^N} \operatorname{div}_y(A \nabla_x u_0)\, dy = 0$

$$\int_{[0,1]^N} \operatorname{div}_y(A \nabla_x u_0)\, dy = \int_{[0,1]^N} A \nabla_x u_0 \, dy$$

$$= 0, \qquad \text{since } A, \nabla_x u_0 \text{ periodic in } y.$$

$\Rightarrow$ u_1 well-def. by (2').

Since $\nabla_x u_0$ indep. of y and A indep. of x: look for solution u_1 in the form

$$u_1(x,y) = -\sum_{j=1}^{N} \chi_j(y) \frac{\partial u_0}{\partial x_j}(x) = -\chi \cdot \nabla_x u_0$$

$\rightsquigarrow$ Problem for χ_j :

$$\left.\begin{aligned} -\operatorname{div}_y(A\nabla_y\chi_j) &= \operatorname{div}_y(Ae_j) \\ \chi_j \text{ is } 1\text{-periodic} \end{aligned}\right\}$$

$\sum_{i=1}^{N} \frac{\partial a_{ij}}{\partial y_i}$

"cell problem"

solvability as above.

Finally, consider (3'):

Well-posedness:

$$0 \overset{!}{=} \int_{[0,1]^N} \Big(f + \big(\operatorname{div}_x A\nabla_y + \underbrace{\operatorname{div}_y A\nabla_x}_{=0}\big)u_1 + \operatorname{div}_x(A\nabla_x u_0)\Big)\,dy$$

$$= \int_{[0,1]^N} f\,dy + \int_{[0,1]^N} \operatorname{div}_x\big(A\nabla_y u_1 + A\nabla_x u_0\big)\,dy$$

$$\Longleftrightarrow \quad f = -\int_{[0,1]^N} \operatorname{div}_x \underbrace{\Big(-\sum_j A\nabla_y\chi_j \frac{\partial u_0}{\partial x_j} + A\nabla_x u_0\Big)}\,dy$$

$$\left(\begin{aligned} &= -A\nabla_y\chi^T\nabla_x u_0 + A\nabla_x u_0 \\ &= (A - A\nabla_y\chi^T)\nabla_x u_0 \end{aligned}\right)$$

$$= -\operatorname{div}_x\Big[\underbrace{\int_{[0,1]^N}(A - A\nabla_y\chi^T)\,dy}_{=:A_0}\cdot\nabla_x u_0\Big]$$

$$= -\operatorname{div}_x(A_0\nabla_x u_0)$$

"Homogenised equation"

Summary:

- Asymptotic expansion: $u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \dots$
- Cell problem: $\left.\begin{aligned} -\operatorname{div}_y(A\nabla_y \chi_j) &= \operatorname{div}_y(A e_j) \\ \chi_j \text{ is } &1\text{-periodic} \end{aligned}\right\}$
- Homogenised matrix: $A_0 = \int_{[0,1]^N} (A - A\nabla_y \chi^T)\, dy$

Justification:

Aim: prove that $\| u_\varepsilon - u_0 \|_{L^\infty(\Omega)} \xrightarrow{\varepsilon \to 0} 0$

Define $z_\varepsilon := u_\varepsilon - u_0 - \varepsilon u_1(\cdot, \frac{\cdot}{\varepsilon}) - \varepsilon^2 u_2(\cdot, \frac{\cdot}{\varepsilon})$. Then

$$\begin{aligned}
-\operatorname{div}(A_\varepsilon \nabla z_\varepsilon) = {} & \underbrace{-\operatorname{div}(A_\varepsilon \nabla u_\varepsilon)}_{=f} + \underbrace{\varepsilon^{-2} \operatorname{div}_y (A\nabla_y u_0)}_{=0} \\
& \underbrace{+\ \varepsilon^{-1}\left(\operatorname{div}_y(A\nabla_y u_1) + \operatorname{div}_y(A\nabla_x u_0) + \operatorname{div}_x(A\nabla_y u_0)\right)}_{=0} \\
& \underbrace{+ \operatorname{div}_y(A\nabla_y u_2) + \operatorname{div}_y(A\nabla_x u_1) + \operatorname{div}_x(A\nabla_y u_1) + \operatorname{div}_x(A\nabla_x u_0)}_{=-f} \\
& + \varepsilon\left(\operatorname{div}_y(A\nabla_x u_2) + \operatorname{div}_x(A\nabla_y u_2) + \operatorname{div}_x(A\nabla_x u_1)\right) \\
& + \varepsilon^2 \operatorname{div}_x(A\nabla_x u_2) \\
= {} & \varepsilon \Big[\operatorname{div}_y(A\nabla_x u_2) + \operatorname{div}_x(A\nabla_y u_2) + \operatorname{div}_x(A\nabla_x u_1) \\
& \quad + \varepsilon \operatorname{div}_x(A\nabla_x u_2) \Big] (x, \tfrac{x}{\varepsilon})
\end{aligned}$$

$$=: \varepsilon r_\varepsilon$$

From definition of u_2:

If A, f, Ω are C^∞, then u_2 smooth in x, y

$\Rightarrow \|r_\varepsilon\|_\infty$ bounded uniformly in ε.

In addition on $\partial\Omega$:

$$Z_\varepsilon\big|_{\partial\Omega} = \left(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2\right)\big|_{\partial\Omega}$$

$$= -\varepsilon\left(u_1 + \varepsilon u_2\right)\big|_{\partial\Omega}$$

$$\Rightarrow \quad \|Z_\varepsilon\|_{L^\infty(\partial\Omega)} \leq C\varepsilon$$

Maximum principle $\Rightarrow \|Z_\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon$

$\square$